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# The structure of the invariants of perfect Lie algebras II

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**Abstract**
The structure of the invariants of perfect Lie algebras with nontrivial centre is analysed. It is shown that if the radical $\mathfrak{r}$ of a semidirect sum $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ of a semisimple and a nilpotent Lie algebra has a one-dimensional centre, then the defining representation $R$ contains a copy of the trivial representation $D_0$ of $\mathfrak{s}$. Using this fact, a criterion can be deduced to eliminate variables and to characterize the Casimir operators of an algebra $\mathfrak{g}$ by means of certain subrepresentations $R'$ of $R$. For rank 1 Levi subalgebras $\mathfrak{s}$, all representations leading to perfect Lie algebras with a radical isomorphic to a Heisenberg Lie algebra are determined. We prove that the number of Casimir operators of such an algebra is fixed for any dimension, and moreover that they can be explicitly computed by means of determinantal formulae obtained from the brackets of the algebra. For rank $n \geqslant 2$ Levi part a stabilization result of this nature is also given.

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## 1. Introduction

Among perfect Lie algebras (i.e. algebras coinciding with their derived subalgebra), semisimple algebras are without doubt the most studied and best understood class in the physical literature. The problem of characterizing their representations and their Casimir operators constitutes nowadays a classical result, where the physical motivation played a decisive role [1–3]. The pioneering work of Racah on the group theory and their applications to spectroscopy contributed to make Lie algebras an essential tool of modern physics, and opened new and alternative approaches to many important physical questions. The branching rules and tensor product decompositions of representations are of great importance in the classification of electron configurations, determination of quantum numbers of a system and classification of particles or nuclear rotational states [4–7]. Most of these techniques are based on the structure of semisimple Lie algebras, but in many problems, such as the combination of relativistic invariance and interaction symmetry [8] or generalization of the Holstein–Primakoff boson formalism [9], this class is not sufficient and we find symmetries described by nonsemisimple Lie algebras. For these no structural theory is known, and the characterization of representation indices and eigenvalues of the Casimir operators is a difficult problem which always forces to concentrate on specific algebras. Even the Levi decomposition of a Lie algebra does not simplify the computation of the Casimir invariants, as could be believed, and algebras of this type having only constant functions as invariants can be found [10]. However, if the algebra is perfect (non-semisimple) some generic results analogous to those known for semisimple algebras can be obtained considering only the Levi part and the defining representation $R$ of the algebra $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$, and under some additional assumptions quite effective bounds for the number of Casimir operators can be deduced [11]. These algebras are also distinguished by the fact that any invariant can be reduced, after symmetrization, to a classical Casimir operator.

In this paper we continue with the analysis outlined in [11] for perfect Lie algebras $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$, and focus on Lie algebras of this type whose defining representation $R$ contains copies of the trivial representation. Although most of the formulae given in [11] are still valid for this case, the case where $R$ contains copies of the trivial representation $D_0$ is of interest. Indeed, we will see that for radicals (which are nilpotent) whose centre is one dimensional, $R$ necessarily contains a copy of $D_0$. This makes from perfect Lie algebras with a one-dimensional centre an exception which is physically relevant since, for example, the Carroll algebra [12] or the $wsp(2, \mathbb{R})$ Lie algebras used in microscopic theory of nuclear collective motions [13] are special cases of this algebra type. The most relevant case is where the radical is isomorphic to a Heisenberg Lie algebra, which leads naturally to the problem of determining all the representations of semisimple Lie algebras $\mathfrak{s}$ such that a perfect Lie algebra $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ with $\mathfrak{r}$ isomorphic to the Heisenberg Lie algebra exists. For rank 1 Levi subalgebras the problem is completely solved, and it is shown that the number of Casimir operators does not depend on the dimension of $\mathfrak{g}$. In particular, both real forms $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3)$ of $\mathfrak{sl}(2, \mathbb{C})$ can be treated simultaneously. Moreover, we give a determinantal formula to determine explicitly the Casimir operators without being forced to solve differential equations or use other alternative methods. This formula is deduced from a specific expansion of the commutator matrix $A(\mathfrak{g})$ comprising the brackets of the algebra. We also show that for the general case of $\mathrm{rank}(\mathfrak{s}) \geqslant 2$, we also obtain a stabilization result for the number $\mathcal{N}(\mathfrak{g})$ of Casimir operators. The result is a direct consequence of the structure of the corresponding commutator matrix $A(\mathfrak{g})$ of perfect Lie algebras having the Heisenberg radical. However, even if the number of invariants can be determined explicitly, there is *a priori* no obvious generic method to compute them in an effective and easy manner.

Unless otherwise stated, any Lie algebra $\mathfrak{g}$ considered in this work is indecomposable and is defined over the field $\mathbb{R}$ of real numbers. We adopt a convention whereby the nonwritten brackets are either zero or obtained by antisymmetry. We also use the Einstein summation convention. Abelian Lie algebras of dimension $m$ will be denoted by $mL_1$.

## 2. Perfect Lie algebras with one-dimensional centre

As told before, we will focus on the invariants of a specific class of algebras, called perfect, which are, in some sense, a natural generalization of semisimple Lie algebras, but allow additional properties such as the existence of central elements.

**Definition 1.** *A Lie algebra* $\mathfrak{g}$ *is called perfect if* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

We recall briefly the standard technique to compute the invariants, in particular the Casimir operators of an algebra [14]. If $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{g}$ and $\left\{C_{ij}^k\right\}$ the structure constants over this basis, we represent $\mathfrak{g}$ in the space $C^\infty(\mathfrak{g}^*)$ by means of the differential operators

$$\widehat{X}_i = -C_{ij}^k x_k \frac{\partial}{\partial x_j} \tag{1}$$

where $[X_i, X_j] = C_{ij}^k X_k$ $(1 \leqslant i < j \leqslant n)$. The operators $\widehat{X}_i$ satisfy the brackets $[\widehat{X}_i, \widehat{X}_j] = C_{ij}^k \widehat{X}_k$ and constitute a representation of $\mathfrak{g}$. An analytic function $F \in C^\infty(\mathfrak{g}^*)$ is called an invariant of $\mathfrak{g}$ if and only if it is a solution of the system:

$$\{\widehat{X}_i F = 0, 1 \leqslant i \leqslant n\}. \tag{2}$$

If $F$ is a polynomial, then it corresponds to a classical Casimir operator, after symmetrization. The system (2) can also have solutions which are not polynomials, in which case we call it a 'generalized Casimir invariant' by analogy with the classical case. Solutions of this type can also be used to label irreducible representations of $\mathfrak{g}$. If (2) has no solutions at all (as happens, for example, for the two-dimensional solvable affine algebra $\mathfrak{r}_2$ or other large classes of solvable Lie algebras [15–18]) we say that the invariants of the coadjoint representation $ad^*$ are trivial. The particular structure of the systems associated with Lie algebras allows us to apply the known techniques from the theory of partial differential equations to compute the cardinal $\mathcal{N}(\mathfrak{g})$ of a maximal set of functionally independent solutions in terms of the brackets of the algebra $\mathfrak{g}$ over a given basis

$$\mathcal{N}(\mathfrak{g}) := \dim \mathfrak{g} - \sup_{x_1,\ldots,x_n} \left\{\operatorname{rank}\left(C_{ij}^k x_k\right)_{1 \leqslant i < j \leqslant \dim \mathfrak{g}}\right\} \tag{3}$$

where $A(\mathfrak{g}) := \left(C_{ij}^k x_k\right)$ is the matrix which represents the commutator table of $\mathfrak{g}$ over the basis $\{X_1, \ldots, X_n\}$ [19]. Evidently this quantity constitutes an invariant of $\mathfrak{g}$ and does not depend on the particular choice of basis.

As an illustrative example of this analytic procedure to compute the invariants and of the class of Lie algebras, we are interested in here to consider the ten-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \overrightarrow{\oplus}_{D_{\frac{3}{2}} \oplus D_{\frac{1}{2}} \oplus D_0} (\mathfrak{h}_1 \oplus 4L_1)$ given by the commutator matrix

$$A(\mathfrak{g}) = \begin{pmatrix} 0 & 2x_2 & -2x_3 & 3x_4 & x_5 & -x_6 & -3x_7 & x_8 & -x_9 & 0 \\ -2x_2 & 0 & x_1 & 0 & 3x_4 & 2x_5 & x_6 & 0 & x_8 & 0 \\ 2x_3 & -x_1 & 0 & x_5 & 2x_6 & 3x_7 & 0 & x_9 & 0 & 0 \\ -3x_4 & 0 & -x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_5 & -3x_4 & -2x_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_6 & -2x_5 & -3x_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3x_7 & -x_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_8 & 0 & -x_9 & 0 & 0 & 0 & 0 & 0 & x_{10} & 0 \\ x_9 & -x_8 & 0 & 0 & 0 & 0 & 0 & -x_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

where the entry $a_{ij}$ corresponds to the bracket $[X_i, X_j]$ over the basis $\{X_1, \ldots, X_{10}\}$. Observe that $\mathfrak{g}$ is indecomposable with a centre generated by $X_{10}$. Moreover $\mathfrak{g}$ is a perfect Lie algebra. Since rank $A(\mathfrak{g}) = 8$, we obtain two invariants, one of which is automatically $I_1 = x_{10}$ for corresponding to a central element. It can be easily shown that a complete set of invariants are given by $I_1$ and $I_2 = -x_5^2 x_6^2 + 4x_6^3 x_4 + 4x_5^3 x_7 - 18x_5 x_6 x_7 x_4 + 27x_7^2 x_4^2$. The interesting fact about this example is that the invariant $I_2$ does not depend on the variables associated with the Levi part and the variables associated with the Heisenberg Lie algebra (the variable corresponding to the centre already being an invariant $I_1$ of the algebra). This shows a pathology which will turn out to be quite typical of perfect Lie algebras having a one-dimensional centre, and which enables us, in a certain sense, to reduce the study to the case of Heisenberg radicals.

In [11] we began the systematic study of the invariants of (nonsemisimple) perfect Lie algebras, and pointed out their interest, for mainly two reasons: on one hand, they constitute a class of algebras containing the semisimple Lie algebras, which allow the extension of some properties of the latter to the general case; and on the other hand, for perfect Lie algebras the Casimir operators always exist. Moreover, we justified that for perfect Lie algebras we can find a fundamental set of invariants $\mathcal{F}$ formed by the Casimir operators (more precisely their existence, their nature having been analysed earlier [20]). We also searched upper bounds for the quantity $\mathcal{N}(\mathfrak{g})$ when the defining representation of $\mathfrak{g}$ did not contain copies of the trivial representation of $\mathfrak{s}$. These bounds were obtained considering only the representation $R$ of $\mathfrak{s}$ describing the semidirect sum, with independence on the structure of the radical (which has to be nilpotent). The upper bound formulae obtained in [11] also remain valid in this case. The main difference lies in the structure of the radical. In general we cannot infer formulae for the invariants when the centre is nonzero, since the radical is necessarily non-Abelian if $\mathfrak{g}$ is indecomposable, and the exact relation of the brackets in the radical with respect to the number of copies of $D_0$ appearing in $R$ cannot be comprised in an argument which remains valid for all possible cases. However, if the radical is itself decomposable as a nilpotent Lie algebra, a combination of the results in [10, 11] allows a useful insight into the structure of the invariants of these algebras. The main interest for perfect Lie algebras of this type for physical applications corresponds to the case where the centre of the radical is one dimensional. Such algebras appeared first in the classification of kinematical Lie algebras [12], and later in the group theoretical analysis of nuclear collective motions by means of the boson formalism [13].

**Proposition 1.** *Let* $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ *be a perfect Lie algebra such that* $\dim Z(\mathfrak{r}) = 1$. *Then* $R$ *contains a copy of the trivial representation* $D_0$ *of* $\mathfrak{s}$. *Moreover, the function* $I = z$ *(*$z$ *being the variable associated with the generator* $Z$ *of the centre) is always an invariant of* $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$.

**Proof.** Since the representation $R$ acts as a derivation on the radical [21], we have that $[\mathfrak{s}, Z(r)] \subset Z(r)$, and since the centre is one dimensional, generated by an element $Z$, for any $X$ in $\mathfrak{s}$ we have $[X, Z] = \lambda_X Z$ with $\lambda_X \in \mathbb{R}$ (since the Levi part acts on the radical by derivations, and the centre is a characteristic ideal). Now for any pair $X_1, X_2 \in \mathfrak{s}$ we have, by the Jacobi identity

$$[X_1, [X_2, Z]] + [Z, [X_1, X_2]] + [X_2, [Z, X_1]] = [Z, [X_1, X_2]] = 0. \tag{5}$$

This implies that $\lambda_{[X_1,X_2]} = 0$. Since $\mathfrak{s}$ is itself perfect for being semisimple, any element $X$ in $\mathfrak{s}$ can be obtained as a bracket $[X_1, X_2]$ for certain elements $X_1, X_2 \in \mathfrak{s}$. Therefore, (5) implies $[X, Z] = 0$ for all $X$ in $\mathfrak{s}$, thus $R$ contains a copy of the trivial representation and $Z$ generates the centre of $\mathfrak{g}$. □

This result was already stated for rank 1 algebras in [10]. We insist on the fact that the assertion follows from the dimension of the centre and the fact that semisimple Lie algebras are themselves perfect. Moreover, even if the centre of the resulting algebra is nontrivial, the algebra is indecomposable whenever the radical is indecomposable as nilpotent Lie algebra. We further observe that the conclusion is false for a centre of dimension $\geqslant 2$. Take for example, the nine-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{2D_1}\mathfrak{r}$, where $\mathfrak{r}$ is given by the brackets $[X_4, X_5] = 2X_9, [X_4, X_6] = X_8$ and $[X_5, X_6] = 2X_9$ over the basis $\{X_4, \ldots, X_9\}$. The radical has thus a three-dimensional centre generated by $\{X_7, X_8, X_9\}$, but the representation describing the semidirect product does not contain copies of $D_0$ (moreover the centre of

$\mathfrak{g}$ reduces to zero). Thus the case of one-dimensional centres is exclusive of the case of representations containing a copy of the trivial representation. From the physical point of view, this is also the most interesting case of perfect Lie algebras (of this type), due to the central role that the Heisenberg Lie algebra plays in applications.

## 3. Embedding of Heisenberg algebras into perfect Lie algebras

Lie algebras which are the semidirect product of a Heisenberg $\mathfrak{h}_n$ of dimension $(2n+1)$ and a semisimple Lie algebra $\mathfrak{s}$ have shown their importance in the microscopic theory of nuclear collective motions [13]. In [22] it was shown that the Casimir operators of the semidirect sums $\mathfrak{h}_n \overrightarrow{\oplus} \mathfrak{u}(n)$ and $\mathfrak{h}_n \overrightarrow{\oplus} \mathfrak{sp}(2n, \mathbb{R})$ can be obtained from the Perelomov–Popov formulae by using a realization of the algebra by means of shift operators. The key fact of this method is the relation between the number of boson operators of the Heisenberg part $\mathfrak{h}_n$ and the rank of the semisimple part, which allows us to transform the bases in order to adapt them to the operators formulae [13, 23].

Rather than using a fixed basis to denote the $(2n+1)$-dimensional Heisenberg Lie algebra $\mathfrak{h}_n$, in what follows we will use the fact that this algebra is completely determined by the property of being the only nilpotent Lie algebra having a one-dimensional centre and whose derived subalgebra $[\mathfrak{h}_n, \mathfrak{h}_n]$ coincides with this centre. This will imply that we can always find a basis $\{X_1, \ldots, X_{2n+1}\}$ such that the centre is generated by $X_{2n+1}$ and the only nonzero brackets are $[X_i, X_{2n+1-i}]$. It is convenient to distinguish two cases: either the radical $\mathfrak{r}$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_m$ for some $m$ or $\mathfrak{r} = \mathfrak{h}_m \oplus \mathfrak{r}'$ (observe moreover that if $R$ contains more than one copy of $D_0$ and the radical is the Heisenberg algebra, $\mathfrak{g}$ is not perfect anymore[1]). We are mainly interested in the first case, which turns out to be the most interesting for applications. We will see that radicals isomorphic to Heisenberg algebras impose some restrictions on the admissible representations. Concerning the second case, we will develop a criterion that eliminates some variables and reduces the problem to the situation of Abelian radicals studied in [11].

**Proposition 2.** *Let $\mathfrak{g}_m = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{h}_m$ be a perfect Lie algebra. Then all algebraically independent Casimir operators of $\mathfrak{g}_m$ other than the invariant corresponding to the generator of the centre depend on all the variables of $\mathfrak{g}_m$.*

The proof follows from system (2) and the following observations concerning the structure of the algebra. If $\{X_1, \ldots, X_q, \ldots, X_{q+2m+1}\}$ is a basis of $\mathfrak{g}$ such that $\{X_1, \ldots, X_q\}$ is a basis of $\mathfrak{s}$ and $\{X_{q+1}, \ldots, X_{2m+1+q}\}$ a basis of $\mathfrak{h}_m$ such that the centre is generated by $X_{2m+1+q}$, then any nonconstant invariant $F$ of $\mathfrak{g}$ different from $I = x_{2m+1+q}$ satisfies

$$\frac{\partial F}{\partial x_{2m+1+q}} \neq 0. \tag{6}$$

Otherwise the structure of the radical would imply that

$$\frac{\partial F}{\partial x_{q+t}} = 0 \qquad 1 \leqslant t \leqslant 2m \tag{7}$$

and since $\mathfrak{g}$ is perfect and $R$ does not contain further copies of $D_0$, for any $X_{q+t}$ we can find $X_{i_0} \in \mathfrak{s}$, $X_{q+s_0} \in \mathfrak{h}_m$ such that $\left[X_{i_0}, X_{t+s_0}\right] = X_{q+t}$. This shows that the equations

$$\widehat{X}_{q+t}.F := -C_{q+t,j}^{k+q} x_{k+q} \frac{\partial F}{\partial x_j} = 0 \qquad 1 \leqslant t \leqslant 2m \tag{8}$$

[1] If $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{h}_m$ is perfect and $R$ contains more than one copy of $D_0$, the structure of the radical implies that $[\mathfrak{h}_m, \mathfrak{h}_m] = Z(\mathfrak{h}_m)$, which is one dimensional. Therefore, there exists an element $X \in \mathfrak{h}_m$ not lying in the centre such that $[\mathfrak{s}, X] = 0$, but this implies that $X \notin [\mathfrak{g}, \mathfrak{g}]$, which contradicts the perfectness of $\mathfrak{g}$.

would be satisfied only if $F = \text{const}$. Similarly, if for an invariant $F$ we have

$$\frac{\partial F}{\partial x_{q+t_0}} = 0 \tag{9}$$

for some $t_0 \in \{1..2m\}$, consider all the basis elements $X_{i_k} \in \mathfrak{s}$ and $X_{q+j_p} \in \mathfrak{h}_m$ such that

$$\left[X_{i_k}, X_{q+j_p}\right] = C_{i_k, j_p+q}^{q+t_0} X_{q+t_0} \neq 0. \tag{10}$$

The corresponding equations $\widehat{X}_{i_k}.F = 0$ show that $\frac{\partial F}{\partial x_{q+jp}} = 0$ for all these $j_p$. It will also follow from the equations $\widehat{X}_{q+j_p}.F = 0$ that $\frac{\partial F}{\partial x_{i_k}} = 0$ for some indices $i_k$. Repeating the argument for all other variables $x_{q+j_p}$ it will follow after some iteration that

$$\frac{\partial F}{\partial x_i} = 0 \qquad 1 \leqslant i \leqslant q. \tag{11}$$

This is a consequence of the structure of the weight diagram of $R$ (which contains only one copy of $D_0$) and the action of the operators associated with the root system of $\mathfrak{s}$ [7, 24]. Therefore, the invariant $F$ would be independent of the variables associated with the Levi subalgebra, and from the structure of the Heisenberg radical it would follow at once that $F = \text{const}$.

This result shows that, even if the Lie algebra has a more elementary nature for having a nontrivial centre, the Casimir operators are much more difficult to determine, since no reduction of elimination of variables of the system (2) can be obtained. In particular, no lower bounds for the number of Casimir operators can be obtained for this case. These algebras can thus be compared with the special affine Lie algebras $\mathfrak{sa}(n, \mathbb{R})$ [25], and whose direct determination of the invariant is a quite elaborate problem (a determinantal argument based on extensions of Lie algebras has been developed in [26] to compute these operators).

We now prove that under some circumstances, the methods of [11] can be adapted for the case where the radical is the direct sum of Heisenberg and Abelian algebras. Recall that for a semisimple Lie algebra $\mathfrak{s}$ the normalization index $n(\mathfrak{s})$ is defined as the minimal positive integer such that if $R$ is an irreducible representation of $\mathfrak{s}$ of dimension $\dim R := \dim(\mathfrak{s}) + n(\mathfrak{s})$, then all Casimir operators of $\mathfrak{s} \overrightarrow{\oplus}_R (\dim R) L_1$ are obtained from the representation matrix $\rho_R(\mathfrak{s})$. This index can be used to deduce a sufficiency criterion to characterize the Casimir operators of an algebra by a certain representation of the Levi part.

**Proposition 3.** *Let $\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{h}_m$ be a perfect Lie algebra. Then for any representation $R'$ such that $\dim R' > n(\mathfrak{s}) + \dim(\mathfrak{s})$, the number of Casimir operators of the (perfect) Lie algebra $\mathfrak{g}' = \mathfrak{s} \overrightarrow{\oplus}_{R \oplus R'} (\mathfrak{h}_m \oplus (\dim R') L_1)$ is given by*

$$\mathcal{N}(\mathfrak{g}') = \dim R' - \text{rank}_{\rho(R')}(\mathfrak{s}) + 1. \tag{12}$$

*Moreover, the Casimir operators of $\mathfrak{g}'$ do not depend on the variables associated with $\mathfrak{s}$ and $\mathfrak{h}_m$ (up to the invariant associated with the generator of the centre).*

**Proof.** Over a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ the system (2) giving the invariants can be written as

$$\begin{pmatrix} A(\mathfrak{s}) & \rho_{R_1}(\mathfrak{s}) & \rho_{R_2}(\mathfrak{s}) & 0 \\ -\rho_{R_1}^T(\mathfrak{s}) & 0 & 0 & 0 \\ -\rho_{R_2}^T(\mathfrak{s}) & 0 & \widetilde{A}(\mathfrak{h}_m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{pmatrix} = 0 \tag{13}$$

where $\widetilde{A}(\mathfrak{h}_m)$ is the submatrix of $A(\mathfrak{h}_m)$ obtained by deleting the column and row corresponding to the generator of the centre.

Since by assumption $\dim R_1 > \dim(s) + n(\mathfrak{s})$, the subsystem

$$\rho_{R_1}(\mathfrak{s}) \left( \frac{\partial F}{\partial x_i} \right)^T_{1 \leqslant i \leqslant \dim(\mathfrak{s})} = 0 \tag{14}$$

which shows that $\frac{\partial F}{\partial x_i} = 0$ for $1 \leqslant i \leqslant \dim(\mathfrak{s})$ [11]. Since the radical is the Heisenberg Lie algebra, we can extract the subsystem

$$A(\mathfrak{h}_m) \left( \frac{\partial F}{\partial y_i} \right)^T_{1 \leqslant i \leqslant \dim(\mathfrak{h}_m)} = 0 \tag{15}$$

proves that

$$\frac{\partial F}{\partial y_i} = 0 \qquad 1 \leqslant i \leqslant \dim(\mathfrak{h}_m) \tag{16}$$

since the only brackets of $\mathfrak{h}_m$ are of the form $[Y_i, Y_{\dim(\mathfrak{r})-i}] = Y_{\dim(\mathfrak{r})}$. Therefore, the invariants of the algebra are given by

$$\rho_{R_1}(\mathfrak{s}) \left( \frac{\partial F}{\partial z_i} \right)^T_{1 \leqslant i \leqslant \dim R_1} = 0 \tag{17}$$

where $\left\{ Z_1, \ldots, Z_{\dim(R_1)} \right\}$ is a basis of the representation $R_1$. Thus the Casimir operators are exactly those of the Lie algebra $\mathfrak{s} \overrightarrow{\oplus}_{R_1} (\dim R_1) L_1$ to which $I = y_{\dim(\mathfrak{r})}$ is added for being a central element. □

In particular the actual number of invariants for these algebras is considerably lower than the value obtained using the formulae of [11]. We remark that this kind of elimination of variables is not possible if the perfect Lie algebra has Abelian radical, or the defining representation does not contain copies of the trivial representation of the Levi part. This fact makes the analysis of perfect algebras with one-dimensional centre of particular interest. An interesting and physically notable problem arises from this context

**Problem.** Given a semisimple Lie algebra $\mathfrak{s}$, for any $n \geqslant 1$ does there exist a (nontrivial) representation $R$ of $\mathfrak{s}$ such that the Lie algebra

$$\mathfrak{g} := \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{h}_m \tag{18}$$

is perfect?

Special cases have already been treated in the literature, such as in the cited $wu(n)$ and $wsp(2n, \mathbb{R})$ algebras [13, 22], subalgebra chains obtained when analysing branching rules [9] or the classifications in low dimensions [21]. A general solution seems however difficult to find, due to the complexity of extracting all the representations of semisimple Lie algebras which are compatible with the brackets defining the Heisenberg Lie algebra (at least for the exceptional case). Moreover, depending on the semisimple Lie algebra, for the first values of $n$ the problem may not have solution. Take, for instance, the Lie algebra $\mathfrak{su}(3)$. Since the fundamental representation is of degree three, there is no perfect Lie algebra $\mathfrak{g}$ having $\mathfrak{su}(3)$ as the Levi part and such that the radical is isomorphic to the three (respectively, five) dimensional Heisenberg Lie algebra $\mathfrak{h}_1$ (respectively, $\mathfrak{h}_2$). The first value for which such an algebra exists is $n = 3$, with the representation $3 \oplus \overline{3} \oplus 1$ (this algebra appears naturally from the case $wu(3)$ of [13]). In the following two sections we will prove that the problem has a solution, for all $n$, when we take the rank 1 Levi subalgebra. Further it will be shown that once a value of $n$ has been found, we obtain a stabilization result for the number of invariant operators.

## 4. Levi subalgebras $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R}), \mathfrak{so}(3)$

In this section, we analyse the perfect Lie algebras with the Heisenberg $\mathfrak{h}_m$ whose Levi subalgebra $\mathfrak{s}$ is of rank 1. This case will show its interest, since the number of Casimir operators can be determined explicitly regardless of the dimension. Moreover, these operators can be computed by means of a generalization of a determinantal method introduced in [26], and avoids completely the methods of differential equations. If $\mathfrak{g}$ is perfect and $\dim(Z(\mathfrak{g})) = 1$, then $R$ contains necessarily a copy of the trivial representation $D_0$ of the Levi part, as we have seen before (to the author's knowledge, Lie algebras like these entered physics for the first time with the Carroll Lie algebra of Bacry and Lévy-Leblond [12]). Since the classification of the irreducible representations of the complex simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ provides real representations of the normal form $\mathfrak{sl}(2, \mathbb{R})$ and complex representations of the compact form $\mathfrak{so}(3)$, the embedding problem can be treated simultaneously for both cases. We detail the embeddings for the algebra $\mathfrak{sl}(2, \mathbb{R})$, the case of $\mathfrak{so}(3)$ being completely analogue.

Let $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_{R\oplus D_0} \mathfrak{r}$ be a perfect Lie algebra, where $R$ does not contain further copies of $D_0$.

**Proposition 4.** *Let $\mathfrak{sl}(2, \mathbb{R}) \overrightarrow{\oplus}_{R\oplus D_0} \mathfrak{r}$ be a perfect Lie algebra. If $R = \sum_{k=1}^{p} D_{l_k}$ with $l_k \in \mathbb{N}$ for all $k$, then $\mathfrak{r}$ can be isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_m$ only if $R = \sum_{k=1}^{\frac{p}{2}} \left(D_{l_k} \oplus D_{l_k}\right)$.*

**Proof.** Suppose first that $R = \sum_{j=1}^{k} D_{l_j} \oplus D_0$ with $\dim R = 2m + 1 = \dim \mathfrak{h}_m$. Let $\{Y_1, \ldots, Y_{2m+1}\}$ be a basis of the Heisenberg Lie algebra, where the nonzero brackets are given by

$$[Y_i, Y_{2m+1-i}] = a_{i,2m+1-i} Y_{2m+1}. \tag{19}$$

Since $X_1$ acts diagonally on the elements of $\mathfrak{h}_m$, we have $[X_1, Y_i] = \lambda_i Y_i$ for $1 \leqslant i \leqslant 2m + 1$. Applying the Jacobi identity to such a pair and $X_1$ we get

$$[X_1, [Y_i, Y_{2m+1-i}]] + [Y_{2m+1-i}, [X_1, Y_i]] - [Y_i, [X_1, Y_{2m+1-i}]] = 0 \tag{20}$$

which shows that $\lambda_i + \lambda_{2m+1-i} = 0$ for all $i$ since $[X_1, [Y_i, Y_{2m+1-i}]] = 0$. This implies that the number of integer spin representations $D_l$ in the decomposition of $R$ must be even, in order to obtain the brackets corresponding to the vectors associated with the zero weight. For a given $l$ the action of $\mathfrak{sl}(2, \mathbb{R})$ is given by

$$[X_1, Y_{i+1}] = (2l - 2i) Y_{i+1} \qquad i \in \{0..2l\} \tag{21}$$

$$[X_2, Y_{i+1}] = (2l + 1 - i) Y_i \qquad i \in \{1..2l\} \tag{22}$$

$$[X_3, Y_{i+1}] = (i + 1) Y_{i+2} \qquad i \in \{0..2l - 1\}. \tag{23}$$

Therefore the zero weight corresponds to the vector $Y_{l+1}$ for each representation $D_l$. Now the brackets $[Y_{l+1}, Y_{l+2}] = 0$ since $\lambda_{l+1} + \lambda_{l+2} \neq 0$. Considering the Jacobi condition

$$[X_2, [Y_{l+1}, Y_{l+2}]] + [Y_{l+2}, [X_2, Y_{l+1}]] - [Y_{l+1}, [X_2, Y_{l+2}]] = 0 \tag{24}$$

we have that $[Y_l, Y_{l+2}] = 0$. Using induction on $0 \leqslant k \leqslant l$, we get

$$[X_2, [Y_{l+1-k}, Y_{l+2+k}]] + [Y_{l+2+k}, [X_2, Y_{l+1-k}]] - [Y_{l+1-k}, [X_2, Y_{l+2+k}]] = 0 \tag{25}$$

from which we deduce that

$$[Y_{l-k}, Y_{l+2+k}] = 0 \qquad 0 \leqslant k \leqslant l - 1 \tag{26}$$

that is the brackets between the vectors corresponding to a copy $D_l$ are all zero. In particular we get that $[Y_1, Y_{2l+1}] = 0$, which shows that if the representation $D_l$ appears in the decomposition of $R$, then another copy with the same maximal weight must appear. It suffices therefore to consider pairs $(D_l, D_l)$ of representations with the same maximal weight, since the general case follows from direct sums over $l$. If $\{Y_1, \ldots, Y_{2l+1}\}$ and $\{Y'_1, \ldots, Y'_{2l+1}\}$ are bases of two copies of $D_l$ contained in $R$, then by (19) we must have $[Y_i, Y'_{2l+1-i}] \neq 0$ if the radical is the Heisenberg algebra. We consider the basis

$$\{Y_1, \ldots, Y_{2l+1}, Y'_1, \ldots, Y'_{2l+1}\} := \{X_4, \ldots, X_{2l+4}, X_{2l+5}, \ldots, X_{4l+5}\}$$

of the radical. The only brackets are

$$[X_{3+k}, X_{4l+6-k}] = a_k X_{4l+6} \qquad 1 \leqslant k \leqslant 2l+1 \tag{27}$$

where the coefficients $a_k$ are the solutions of the system

$$2la_1 + a_2 = 0 \tag{28}$$

$$(2l-k)a_{k+1} + (k+1)a_{k+2} = 0 \tag{29}$$

for $1 \leqslant k \leqslant 2l-1$. Without loss of generality we can choose $a_1 = 1$, and solving the system gives the coefficient formula

$$a_k = -\frac{(-1)^k (2l)!}{\Gamma(k)\Gamma(2l+2-k)} \qquad 1 \leqslant k \leqslant 2l+1 \tag{30}$$

where $\Gamma(z)$ is the gamma function. It follows in particular that the coefficients are symmetric, i.e. we have

$$a_j = a_{2l+2-j} \qquad 1 \leqslant j \leqslant l$$

and the only coefficient appearing once is $a_{l+1}$. Since the Jacobi conditions are satisfied, equation (30) gives an embedding of $\mathfrak{h}_{2l+1}$ as the radical of a perfect Lie algebra with describing representation $R = (D_l \oplus D_l) \oplus D_0$. The generalization is straightforward. □

It remains to see what happens for half-integer spin representations. Indeed it suffices to prove it for representations of the type $R = D_{\frac{2m+1}{2}} \oplus D_0$, from which the general result will follow at once.

**Proposition 5.** *For any integer* $2m+1 \geqslant 1$ *there exists a perfect Lie algebra* $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_{\frac{2m+1}{2}} \oplus D_0} \mathfrak{r}$ *having the* $(2m+3)$*-dimensional Heisenberg algebra as radical.*

**Proof.** Let $\{X_1, \ldots, X_{2m+6}\}$ be a basis of $\mathfrak{g}$ such that $\{X_1, X_2, X_3\}$ spans $\mathfrak{sl}(2,\mathbb{R})$, $\{X_4, \ldots, X_{2m+6}\}$ is a basis of $\mathfrak{r}$ and such that $[\mathfrak{sl}(2,\mathbb{R}), X_{2m+6}] = 0$. The action of $\mathfrak{sl}(2,\mathbb{R})$ over $\{X_4, \ldots, X_{2m+5}\}$ is given by

$$[X_1, X_{i+4}] = (2m+1-2i)X_{i+4} \qquad i \in \{0..2m+1\} \tag{31}$$

$$[X_2, X_{i+4}] = (2m+2-i)X_{i+3} \qquad i \in \{1..2m+1\} \tag{32}$$

$$[X_3, X_{i+4}] = (i+1)X_{i+5} \qquad i \in \{0..2m\}. \tag{33}$$

Since $X_1$ acts diagonally and the radical is the Heisenberg Lie algebra, only vectors having opposite weight for the action of $X_1$ can have a nonzero bracket. Therefore, we obtain that

$$[X_{4+i}, X_{2m+5-i}] = \lambda_{i+1} X_{2m+6} \qquad 0 \leqslant i \leqslant m. \tag{34}$$

Applying the Jacobi condition to the triples $\{X_i, X_{4+j}; X_{4+k}\}$ with $i = 2, 3$, we obtain that the coefficients $\lambda_{i+1}$ are solutions of the following system:

$$\lambda_1 + 2m + 1 = 0 \tag{35}$$

$$k\lambda_k + (2m+2-k)\lambda_{k-1} = 0 \qquad 2 \leqslant k \leqslant m. \tag{36}$$

Elementary algebraic manipulation and a recurrence argument allows us to obtain the following formula for the coefficients $\lambda_k$

$$\lambda_{k+1} = \frac{(2m+1)\Gamma(2m+1)(-1)^k}{\Gamma(k+1)\Gamma(2m+2-k)} \qquad 0 \leqslant k \leqslant m \tag{37}$$

All other Jacobi conditions are automatically satisfied, so that (34) proves that the Heisenberg Lie algebra is the radical of $\mathfrak{g}$. □

In the previous results, the gamma function $\Gamma(z)$ appeared in the structure constants of $\mathfrak{r}$ due to the diagonal nature of the element $X_1$ of $\mathfrak{sl}(2,\mathbb{R})$, which forces us to rescale the basis of the radical in order to adapt it to the coefficients of the action $[X_1, \mathfrak{r}]$ and the brackets of the Heisenberg Lie algebra.

In consequence we obtain the general shape of $\mathfrak{sl}(2,\mathbb{R})$-representations leading to irreducible embeddings of the Heisenberg algebra:

**Corollary 1.** *Let* $\mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_R \mathfrak{r}$ *be a perfect Lie algebra with one-dimensional centre. Then* $\mathfrak{r}$ *is isomorphic to the Heisenberg Lie algebra only if*

$$R = \sum_{i=1}^{p} \left(D_{j_i} \oplus D_{j_i}\right) \oplus \sum_{l=1}^{q} D_{\frac{2k_l+1}{2}} \oplus D_0$$

*with* $j_i, k_l \in \mathbb{N}$.

Exactly the same analysis can be applied to the complex irreducible representations $D_j$ of $\mathfrak{so}(3)$, with similar results. In particular, integer spin representations $D_l$ ($l \in \mathbb{Z}$) must appear in pairs, while half-integer representations $D_j$ can appear in single copies in the decomposition of the defining representation $R$. To obtain the corresponding real representations, we use Cartan's theorem [27]. Recall that for the half-integer representations $D_j$ given by the (complex) matrix $A = A_1 + \mathrm{i}A_2$, the real representation is obtained from the real matrix of double size

$$A^{\mathrm{II}} = \left(\begin{array}{c|c} A_1 & -A_2 \\ A_2 & A_1 \end{array}\right). \tag{38}$$

Following [27, 28], these representations are called of second class and denoted by $D_j^{\mathrm{II}}$. If $D_l$ is an integer spin representation, then there exists a change of basis such that the matrices of $D_l$ on the transformed basis are real. These representations are called first class and denoted by $D_l^{\mathrm{I}}$. Therefore, once the complex representations $R_{\mathbb{C}}$ of $\mathfrak{so}(3)$ compatible with the Heisenberg radical have been determined, the real representations $R$ that can be combined with Heisenberg algebras followed by application of the preceding method.

**Corollary 2.** *Let* $\mathfrak{so}(3) \overrightarrow{\oplus}_R \mathfrak{r}$ *be a perfect Lie algebra with one-dimensional centre. Then* $\mathfrak{r}$ *is isomorphic to the Heisenberg Lie algebra only if*

$$R = \sum_{i=1}^{p} \left(D^{\mathrm{I}}_{j_i} \oplus D^{\mathrm{I}}_{j_i}\right) \oplus \sum_{l=1}^{q} D^{\mathrm{II}}_{\frac{2k_l+1}{2}} \oplus D_0$$

*with* $j_i, k_l \in \mathbb{N}$.

## 5. Determinantal formula for the noncentral Casimir operator

As follows from the previous paragraphs, the embedding of Heisenberg Lie algebras as radical of the perfect Lie algebra $\mathfrak{g}$ whose Levi part is of rank 1 is essentially the same for the real forms of $\mathfrak{sl}(2)$. This similarity suggests that the study of the Casimir operators can also be developed simultaneously for both cases. Indeed we will have an interesting and unexpected consequence concerning the Casimir operators of the corresponding perfect Lie algebra $\mathfrak{g}$. This pathology is also exclusive of perfect Lie algebras having a one-dimensional centre, and cannot occur for the type of algebras considered in [10, 11], and more concretely, for perfect algebras having Abelian radicals.

**Proposition 6.** *Let* $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m$ $(m \geqslant 1)$ *be a perfect Lie algebra with one-dimensional centre, where* $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R})$ *or* $\mathfrak{so}(3)$. *Then*

$$\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m) = 2.$$

**Proof.** The commutator matrix of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m$ is of the shape

$$A(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m) = \begin{pmatrix} A(\mathfrak{s}) & \cdots & \rho_R(\mathfrak{s}) \\ \vdots & & \vdots \\ -\rho_R(s)^T & \cdots & A(\mathfrak{h}_m) \end{pmatrix} \tag{39}$$

where we can suppose that the matrix $A(\mathfrak{h}_m)$ corresponding to the radical is

$$A(\mathfrak{h}_m) = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & \alpha_1 z & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_m z & \cdots & 0 & 0 \\ 0 & \cdots & -\alpha_m z & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ -\alpha_1 z & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{40}$$

with $Z$ generating the centre of $\mathfrak{h}_m$ (and $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m$) and the $\alpha_i$ are nonzero. Since rank $A(\mathfrak{h}_m) = 2m$ and rank $A(\mathfrak{s}) = 2$, we easily see that rank $A(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m) \geqslant 2m+1$. Due to the presence of a zero row (corresponding to the centre) and the skew symmetry, it follows that rank $A(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m) = 2m+2$, and the assertion follows from (3). □

Observe further that from the two Casimir operators of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m$, one of them corresponds necessarily to the central element, so that we have only to compute one Casimir invariant. As follows from the discussion of the general case, this second invariant will be dependent on all the variables of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m$. A direct calculation of it constitutes a hard problem for growing $m$, so that a direct approach seems inappropriate. However, this second Casimir operator can be determined explicitly starting from the basis used in the proof of propositions 4 and 5, if we adapt a general method introduced in [26] to perfect Lie algebras of this type.

**Theorem 1.** *Let* $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m$ $(m \geqslant 1)$ *be a perfect Lie algebra with one-dimensional centre, where* $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R})$ *or* $\mathfrak{so}(3)$. *Then the noncentral Casimir operator* $C$ *of* $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m$ *is given by the formula*

$$C^2 = \left| \begin{pmatrix} & & & & & & x_1 \\ & & & & & & x_2 \\ & & & & & & x_3 \\ & & & A(\mathfrak{g}) & & & \frac{1}{2}x_4 \\ & & & & & & \vdots \\ & & & & & & \frac{1}{2}x_{\dim g-1} \\ -x_1 & -x_2 & -x_3 & -\frac{1}{2}x_4 & \cdots & -\frac{1}{2}x_{\dim g-1} & 0 \end{pmatrix} \right|. \tag{41}$$

The proof also follows by induction on $m$ and is technically the same as the proof given in [26] for Abelian radicals. It is based on the reformulation of system (2) in terms of total differential equations, and the fact that the square root of the determinant above comprises the solution to these equations [26, 29]. However, a very important remark must be made in this case, namely that the matrix of (41) does not correspond to a Lie algebraic object (as it was the case analysed in [26]), such as an extension or a deformation of the perfect Lie algebra $\mathfrak{g}$. It is only a formal skew-symmetric linear operator which comprises the information of the solutions of the system (2), and has no obvious interpretation within the frame of representation theory. Inspite of this fact, formula (41) is an economical and easy method to determine the Casimir operator of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{h}_m$ without being forced to solve PDEs, and can easily be deduced from the commutator matrix $A(\mathfrak{g})$ of the algebra.

Consider, for example, the 12-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{D_{\frac{7}{2}}\oplus D_0}\mathfrak{h}_4$. Over the basis $\{X_1,\ldots,X_{12}\}$ the action of $\mathfrak{sl}(2,\mathbb{R})$ is given by equations (31)–(33), while the coefficients $\lambda_k$ of the radicals are, respectively,

$$\lambda_0 = 1 \qquad \lambda_1 = -7 \qquad \lambda_2 = 21 \qquad \lambda_3 = -35.$$

Clearly $\mathcal{N}(\mathfrak{g}) = 2$ and $I = x_{12}$ is an invariant for corresponding to the generator of the centre. The noncentral invariant $C$ is obtained by evaluation of the corresponding determinant (41), and equals

$$\begin{aligned}C = {} & 5x_6^2x_9^2 + 8\left(x_8^3x_6 + x_7^3x_9\right) + 3430(x_1x_4x_{11}x_{12} - x_4x_5x_{10}x_{11}) - 98x_4x_7x_8x_{11}\\ & + 140(x_2x_7x_9x_{12} - x_5x_6x_8x_{11} - x_4x_7x_9x_{10} - x_3x_6x_8x_{12}) + 56\big(x_5x_7^2x_{11}\\ & + x_3x_7^2x_{12} + x_4x_8^2x_{10} - x_2x_8^2x_{12}\big) + 980\left(x_2x_5x_{11}x_{12} - x_3x_4x_{10}x_{12} + x_2x_3x_{12}^2\right)\\ & + 280\left(x_5^2x_9x_{11} + x_4x_6x_{10}^2 + x_3x_5x_9x_{12} - x_2x_6x_{10}x_{12}\right) + 40\left(x_6^2x_8x_{10} + x_5x_7x_9^2\right)\\ & - 16\left(x_6x_7^2x_{10} + x_5x_8^2x_9\right) + 490x_4x_6x_9x_{11} - 3x_7^2x_8^2 - 22x_6x_7x_8x_9\\ & + 245x_1^2x_{12}^2 + 12\,005x_4^2x_{11}^2 + 10x_5x_7x_8x_{10} - 350x_1x_5x_{10}x_{12} - 130x_5x_6x_9x_{10}\\ & - 14x_1x_7x_8x_{12} + 125x_5^2x_{10}^2 + 70x_1x_6x_9x_{12}.\end{aligned}$$

The advantage of this method compared with a direct integration of the corresponding system (2) or other alternative methods is considerable.

## 6. A stabilization result for higher ranks

The preceding results show that for all $n \geqslant 1$ the Heisenberg Lie algebra can be embedded into a perfect Lie algebra with the Levi part $\mathfrak{sl}(2,\mathbb{R})$, respectively, $\mathfrak{so}(3)$. Moreover, the number of Casimir operators is fixed, and the noncentral invariant can be computed by pure algebraic means, using determinantal methods similar to those introduced in [26]. As commented, for higher ranks of the Levi part $\mathfrak{s}$, the problem may have no solution for the first values, and the problem is of interest once the lowest value has been found. It seems improbable that a complete determination of the representations $R$ of $\mathfrak{s}$ compatible with the Heisenberg radical can be found. However, for some special cases we will see that the number $\mathcal{N}(\mathfrak{g})$ of Casimir operators also stabilizes.

Let $\mathfrak{s}$ be a semisimple Lie algebra and $R$ a representation not containing copies of the trivial representation $D_0$ such that

$$\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_{R\oplus D_0}\mathfrak{h}_{\frac{1}{2}\dim R} \tag{42}$$

is perfect. Suppose moreover that $\mathcal{N}(\mathfrak{g}) = q$.

**Theorem 2.** *For any $k \geqslant 1$ the Lie algebra*

$$\mathfrak{g}_k = \mathfrak{s} \overrightarrow{\oplus}_{kR \oplus D_0} \mathfrak{h}_{\frac{k}{2}\dim R} \tag{43}$$

*is perfect and satisfies the identity*

$$\mathcal{N}(\mathfrak{g}_k) = q. \tag{44}$$

**Proof.** The perfectness of the algebras is obvious. We prove the assertion by induction of $k$. For $k = 1$ it is obviously true. Suppose it holds for $k > 1$. Without loss of generality we label the copies of $R$ by $R_1, \ldots, R_k, R_{k+1}$. The commutator matrix of $A(\mathfrak{g}_{k+1})$ is of the shape

$$A(\mathfrak{g}_{k+1}) = \begin{pmatrix} A(\mathfrak{s}) & \rho_{R_1}(\mathfrak{s}) & \cdots & \rho_{R_k}(\mathfrak{s}) & \rho_{R_{k+1}}(\mathfrak{s}) & 0 \\ -\rho^T_{R_1}(\mathfrak{s}) & \widetilde{A}\left(\mathfrak{h}_{\frac{1}{2}\dim R}\right) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ -\rho^T_{R_k}(\mathfrak{s}) & 0 & \cdots & \widetilde{A}\left(\mathfrak{h}_{\frac{1}{2}\dim R}\right) & 0 & 0 \\ -\rho^T_{R_{k+1}}(\mathfrak{s}) & 0 & \cdots & 0 & \widetilde{A}\left(\mathfrak{h}_{\frac{1}{2}\dim R}\right) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{45}$$

where the matrices $\widetilde{A}\left(\mathfrak{h}_{\frac{1}{2}\dim R}\right)$ comprise the commutators of $\mathfrak{h}_{\frac{1}{2}\dim R}$ corresponding to noncentral elements. By assumption, $\mathcal{N}(\mathfrak{g}_k) = q$, so that

$$\operatorname{rank} A(\mathfrak{g}_k) = \dim \mathfrak{s} + k \dim R + 1 - q. \tag{46}$$

Since $A(\mathfrak{g}_k)$ is a submatrix of $A(\mathfrak{g}_{k+1})$, we easily see that $\operatorname{rank} A(\mathfrak{g}_{k+1}) > \operatorname{rank} A(\mathfrak{g}_k)$. Now observe that the submatrix

$$B := \begin{pmatrix} \rho_{R_1}(\mathfrak{s}) & \cdots & \rho_{R_k}(\mathfrak{s}) & \rho_{R_{k+1}}(\mathfrak{s}) \\ \widetilde{A}\left(\mathfrak{h}_{\frac{1}{2}\dim R}\right) & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & \widetilde{A}\left(\mathfrak{h}_{\frac{1}{2}\dim R}\right) & 0 \\ 0 & \cdots & 0 & \widetilde{A}\left(\mathfrak{h}_{\frac{1}{2}\dim R}\right) \end{pmatrix} \tag{47}$$

has maximal rank, due to the fact that the radical of $\mathfrak{g}_{k+1}$ is isomorphic to the Heisenberg Lie algebra. In fact the matrix of the radical is, after a convenient permutation, a diagonal matrix with nonzero entries. It is straightforward to see that we obtain that

$$\operatorname{rank} \begin{pmatrix} A(\mathfrak{s}) & \rho_{R_1}(\mathfrak{s}) & \cdots & \rho_{R_k}(\mathfrak{s}) \\ -\rho^T_{R_1}(\mathfrak{s}) & \widetilde{A}\left(\mathfrak{h}_{\frac{1}{2}\dim R}\right) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -\rho^T_{R_k}(\mathfrak{s}) & 0 & \cdots & \widetilde{A}\left(\mathfrak{h}_{\frac{1}{2}\dim R}\right) \\ -\rho^T_{R_{k+1}}(\mathfrak{s}) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \dim \mathfrak{g}_k - 1 \tag{48}$$

for $k > 1$. Now the adjunction of the last $(\dim R)$ columns of $A(\mathfrak{g}_{k+1})$ increases the rank by $(\dim R) - q + 1$, due to the fact that $\mathcal{N}(\mathfrak{g}_k) = q$ (where the commutator matrix of the semisimple part interacts with the representation) and the maximal rank of matrix (47). This shows that the rank of $A(\mathfrak{g}_{k+1}) = A(\mathfrak{g}_k) + (\dim R)$, and by (3) we obtain

$$\operatorname{rank} A(\mathfrak{g}_{k+1}) = \dim \mathfrak{s} + (k+1)\dim R + 1 - q \tag{49}$$

which shows that $\mathcal{N}(\mathfrak{g}_{k+1}) = q$. □

The interesting fact about this proof is that the rank of $\mathfrak{g}_k$ is determined by that of $\mathfrak{g}_1$, where the submatrix $A(\mathfrak{s})$ corresponding to the semisimple part plays an essential role. For all further adjoined copies, due to the particular structure of the radical, the rank will be obtained by adding the corresponding degree. In this sense we can say that the number of invariants is given by the semisimple part and the representation $R$. It would be interesting to know if a formula for all possible (nonequivalent) representations which express the number of invariants of such perfect algebras exists (for rank $\geqslant 2$, the rank 1 case following from the preceding discussion).

To illustrate the argument of the proof, consider the ten-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{3D_{\frac{1}{2}} \oplus D_0} \mathfrak{h}_3$. Over a basis $\{X_1, \ldots, X_{10}\}$ the commutator matrix $A(\mathfrak{g})$ is given by

$$A(\mathfrak{g}) = \begin{pmatrix} 0 & 2x_2 & -2x_3 & x_4 & -x_5 & x_6 & -x_7 & x_8 & -x_9 & 0 \\ -2x_2 & 0 & x_1 & 0 & x_4 & 0 & x_6 & 0 & x_8 & 0 \\ 2x_3 & -x_1 & 0 & x_5 & 0 & x_7 & 0 & x_9 & 0 & 0 \\ -x_4 & 0 & -x_5 & 0 & x_{10} & 0 & 0 & 0 & 0 & 0 \\ x_5 & -x_4 & 0 & -x_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_6 & 0 & -x_7 & 0 & 0 & 0 & x_{10} & 0 & 0 & 0 \\ x_7 & -x_6 & 0 & 0 & 0 & -x_{10} & 0 & 0 & 0 & 0 \\ -x_8 & 0 & -x_9 & 0 & 0 & 0 & 0 & 0 & x_{10} & 0 \\ x_9 & -x_8 & 0 & 0 & 0 & 0 & 0 & -x_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{50}$$

Since rank $A(\mathfrak{g}) = 8$, we get $\mathcal{N}(\mathfrak{g}) = 2$. The noncentral Casimir operator can also be computed easily using the determinantal method of (41). Observe that $\mathfrak{g}' = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{2D_{\frac{1}{2}} \oplus D_0} \mathfrak{h}_2$ is of dimension eight and has also two invariants (such as $\mathfrak{g}'' = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_{\frac{1}{2}} \oplus D_0} \mathfrak{h}_1$), and that the submatrix of (50) obtained from the first seven and the last columns has rank 7 (the rank of $A(\mathfrak{g}') + 1$, due to the nonzero entries of the eight and ninth rows). This implies that the eight and ninth columns cannot be independent of the first seven, but rather that the rank increases by one. Therefore, we have rank $A(\mathfrak{g}) =$ rank $A(\mathfrak{g}') + 2 =$ rank $A(\mathfrak{g}'') + 4$. Thus the number of invariants is fixed and depends only on that of $A(\mathfrak{g}'')$. However, it is not obvious how to obtain the Casimir operators of $A(\mathfrak{g})$ starting from those of $A(\mathfrak{g}')$.

A direct consequence of this result is the possibility of generalizing some conclusions of [13]: if we define $w_k(n) = \mathfrak{h}_{kn} \overrightarrow{\oplus} \mathfrak{u}(n)$ and $wsp_k(n) = \mathfrak{h}_{kn} \overrightarrow{\oplus} \mathfrak{sp}(2n,\mathbb{R})$ for $k \geqslant 1$, we obtain that

$$\mathcal{N}(w_k(n)) = n \tag{51}$$

$$\mathcal{N}(wsp_k(n)) = n. \tag{52}$$

In view of this, one should expect that the shift operator method used in [13, 22] to compute the Casimir operators of $w(n)$ and $wsp(n)$ in dependence of those of $\mathfrak{u}(n)$ and $\mathfrak{sp}(2n,\mathbb{R})$ can be generalized without problems to obtain the Casimir operators for any $k > 2$. Two questions arise in this context. First, if the underlying representations of $w(n)$ and $wsp(n)$ expressing the semidirect sum are the only compatible with the Heisenberg radical, or if there are others not covered by $w_k(n)$ and $wsp_k(n)$. The other question refers to the fact if for any perfect Lie algebra having the Heisenberg radical the Casimir operators can be deduced by means of a boson operator formalism like that of the preceding algebras. Of special interest is the case of exceptional Levi subalgebras for their role in high energy physics [30]. This case is however more involved due to the less transparent structure of their representations, and the fact that the lowest dimensional perfect Lie algebra of this type appears in dimension 29.

## 7. Conclusions

The analysis of the Casimir operators of perfect Lie algebras with nonzero centre is more difficult than in the centreless case studied in [11], since the structure of the radical plays a more important role than in the algebras studied there. A special case appears when the radical has a one-dimensional centre, which implies automatically that the perfect algebra has itself nontrivial centre (this being false if the radical has a centre of dimension at least two). This fact allows us to obtain a criterion to eliminate the variables of the Levi part and of some parts of the radical, thus to obtain partial stabilization results on the number of Casimir operators. Moreover, in these cases the invariants of the whole algebra are completely determined by some summand of the defining representation $R$ of $\mathfrak{g}$. This fact is typical of perfect algebras having a one-dimensional centre, and leads naturally to analyse the embedding of Heisenberg Lie algebras $\mathfrak{h}_m$ (being characterized by their one-dimensional centre coinciding with the derived subalgebra) into perfect Lie algebras. Physically this is also the most relevant case due to the deep relation of the Heisenberg Lie algebra with the harmonic oscillator. The stabilization result cited above naturally reduces the case where the radical $\mathfrak{r}$ is isomorphic to $\mathfrak{h}_m$. The irreducible embedding of Heisenberg Lie algebras into the perfect Lie algebra whose Levi part is of rank 1 has shown that perfect Lie algebras with one-dimensional centre constitute a special class within perfect algebras, and that their invariants can, under some restrictions, be determined explicitly. It has been pointed out that for the two real forms of $\mathfrak{sl}(2, \mathbb{C})$, the embedding of Heisenberg radicals is essentially the same (up to the doubling of the dimension for the real representations of second class of $\mathfrak{so}(3)$). The constancy of the number of Casimir operators also shows that upper bounds for $\mathcal{N}(\mathfrak{g})$ become quite ineffective, due to the stabilization of the number of Casimir invariants. This is an exclusive phenomenon of nilpotent radicals of the lowest possible nilindex, and does not occur in the perfect algebras of the type analysed in [11]. The corresponding invariants are obtained by pure algebraic methods, starting from the commutator matrix of the algebra. As a consequence, the eigenvalues of representations and other relevant quantities can be obtained without effort from an extended commutator matrix. The interesting observation concerns the expansion problem and the contractions of Lie algebras [31], since it shows that there exist deformations of (decomposable) perfect Lie algebras with Abelian radical that kill almost all Casimir operators. In fact, for the case of rank 1 Levi part analysed, the number of invariants is minimal. One can ask if for higher ranks the (perfect) Lie algebras with Heisenberg radicals also represent the class of Lie algebras having nontrivial Levi decomposition and a minimal number of invariants. A complete answer seems difficult in view of the complexity of characterizing all representations of a semisimple Lie algebra that lead to semidirect sums having this structure. Some important cases have already been studied [22], which show that under certain assumptions the boson realization of Lie algebras allow us to apply the Perelomov–Popov formulae to compute the invariants. The advantage of the boson formalism lies in the possibility of having supplementary manipulation freedom, since the brackets can be expanded, while in the pure representation theoretical approach only the brackets have a significance. This method has however an obvious advantage, namely the possibility of recognizing more easily the representations involved. In this context, it is an interesting problem whether any perfect Lie algebra (with the Heisenberg radical) can be described in a closed form using boson realizations. Independently of its own interest, such a description would be very useful for the analysis of branching rules or the classification of subgroup structures [32, 33]. The stabilization result of section 6 shows that if once a representation is known, further copies of it will not increase the number of Casimir operators of the corresponding algebra. This reduces the problem to the determination of the representations.

Another question that arises here is whether by a recurrence argument the invariants of the algebras $\mathfrak{g}_{k+1}$ (see (43)) can be obtained using the knowledge of the Casimir operators of $\mathfrak{g}_k$.

Another potential application concerns the theory of linear partial differential equations. The determinants (41) show that the solutions of the corresponding system can be evaluated directly from the coefficients of the system, and that the number of independent solutions does not depend on the number of variables. In particular these systems can be seen as deformations of the systems corresponding to the contracted algebras having Abelian radical. Therefore we see the possibility of deforming a linear system to have a minimal number of solutions. It would be interesting to characterize all the systems of PDEs having this property.

Finally, these problems lead to the question whether perfect Lie algebras are the suitable class to obtain intrinsic formulae for the Casimir operators, in analogy to the semisimple case. It seems moreover from the known classifications [21, 34] that the radicals of perfect Lie algebras are necessarily of very low nilpotence indices in order to be compatible with nontrivial representations of semisimple algebras (unless the representation contains a great number of copies of the trivial representation). If this assertion is proved to hold, this would provide us a strong structural restriction that could allow a systematization and classification of perfect Lie algebras and their invariants by pure representation theoretical means.

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